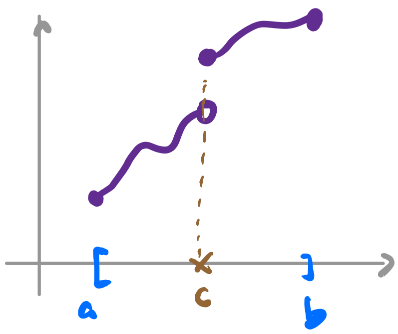
MATH 2050 - Monotone and inverse functions

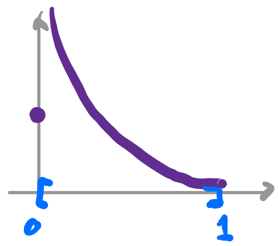
(Reference: Bartle § 5.6)

Q: Consider a function $f : [a,b] \longrightarrow \mathbb{R}$, what kind of "discontinuity" can appear?

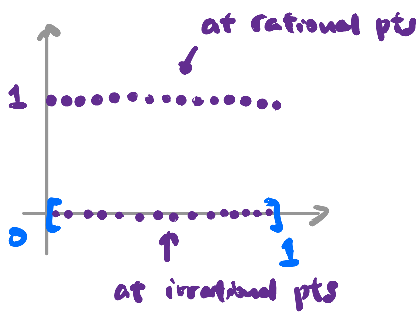
Some examples of discontinuous functions



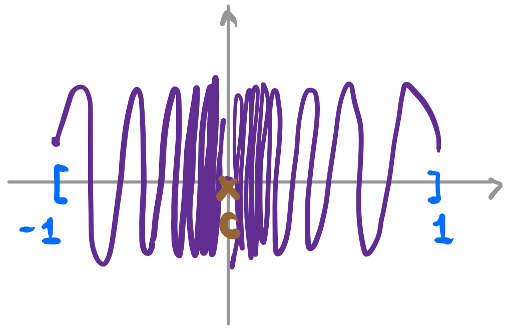
"jump discontinuity"



" ∞ discontinuity"



" everywhere discts "
("densely discrete")



"oscillatory discontinuity"

Q: Classification in general seems out of reach. So what about just for some "simpler" functions?

A: Yes, e.g. for monotone functions.

Def^n: Let $f : A \to \mathbb{R}$. We say that

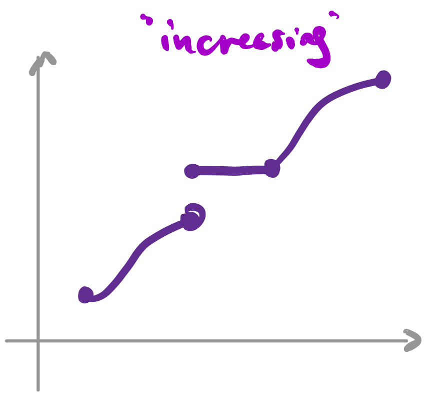
(i) f is (strictly) increasing if the following holds:

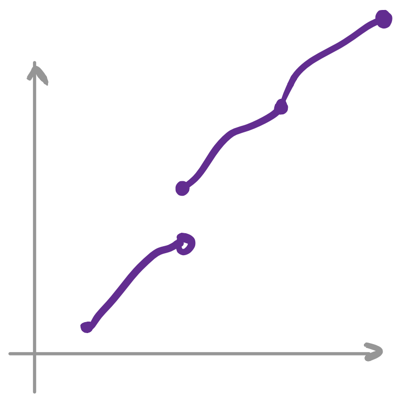
$$x_1, x_2 \in A \ \& \ x_1 \overset{(<)}{\leq} x_2 \implies f(x_1) \overset{(<)}{\leq} f(x_2)$$

(ii) f is (strictly) decreasing if the following holds:

$$x_1, x_2 \in A \ \& \ x_1 \overset{(<)}{\leq} x_2 \implies f(x_1) \overset{(>)}{\geq} f(x_2)$$

(iii) f is (strictly) monotone if it is either (strictly) increasing / decreasing.





GOAL: Monotone functions on $[a,b]$ <u>ONLY</u> have "jump discontinuities".

We shall need the notion of "1-sided limits".

Def^n: Let $f : A \to \mathbb{R}$ and $c \in \mathbb{R}$ is a cluster point of $A \cap (c, \infty)$.

$\lim\limits_{x \to c^+} f(x) = L$ iff $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ s.t.

"right-hand limit"

$|f(x) - L| < \varepsilon$ whenever $x \in A$ and $0 < x - c < \delta$

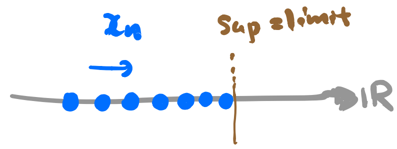
Remark: We can define similarly $\lim\limits_{x \to c^-} f(x) = L$.

Thm: $\lim_{x\to c} f(x) = L \iff \lim_{x\to c^-} f(x) = L = \lim_{x\to c^+} f(x)$

Pf: Exercises.

Recall: MCT : (x_n) increasing & bdd above

$\Rightarrow \lim_{n\to\infty} (x_n) = \sup\{x_n \mid n\in\mathbb{N}\}$.

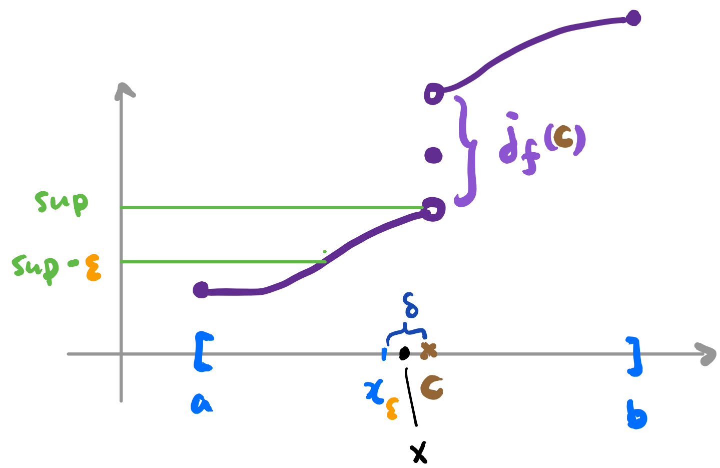


Thm: Let $f : [a,b] \to \mathbb{R}$ be an increasing function.

For any $c \in (a,b)$, we have

$$\lim_{x\to c^-} f(x) = \sup_{x\in[a,c)} f(x) \quad \& \quad \lim_{x\to c^+} f(x) = \inf_{x\in(c,b]} f(x)$$

Picture:



Proof: We just show $\lim_{x\to c^-} f(x) = \sup_{x\in[a,c)} f(x)$. ← exist ∵ bdd above by $f(c)$

Let $\varepsilon > 0$ be fixed but arbitrary.

$\exists\, x_\varepsilon \in [a,c)$ s.t. $\sup_{x\in[a,c)} f(x) - \varepsilon < f(x_\varepsilon)$

Take $\delta := c - x_\varepsilon > 0$. Then, $\forall\, x \in [a,c)$ st $\underline{0 < c - x < \delta}$

we have $x_\varepsilon < x < c$, and hence ($\because f$ increasing)

$$\sup_{x\in[a,c)} f(x) - \varepsilon < f(x_\varepsilon) \le f(x) \le \sup_{x\in[a,c)} f(x)$$

$\blacksquare$

Cor: Same assumption as in Thm. THEN:

$$f \text{ cts at } c \in (a,b) \iff \sup_{x\in[a,c)} f(x) = f(c) = \inf_{x\in(c,b]} f(x)$$

Def^n: Let $f:[a,b]\to\mathbb{R}$ be an increasing function & $c\in(a,b)$.

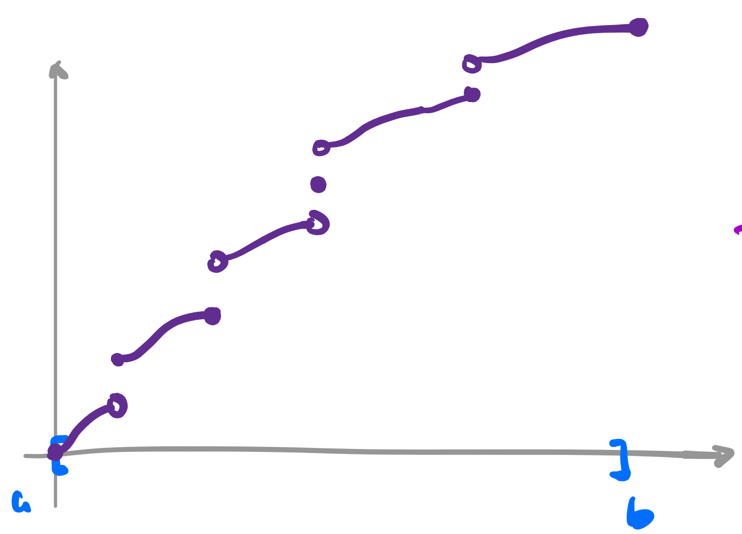
Define the jump of f at c to be

$$j_f(c) := \lim_{x\to c^+} f(x) - \lim_{x\to c^-} f(x)$$

Note: $j_f(c) \geq 0$ and "=" holds $\iff$ f is cts at c

Thm: Let $f:[a,b]\to\mathbb{R}$ be an increasing function.

THEN, the set of $c\in[a,b]$ s.t. f is discontinuous at c is at most countable.



i.e. $\exists$ only at most countably many jump discontinuities for a monotone fcn.

Proof: Denote the set of discontinuity

$$D := \{ c \in (a,b) \mid j_f(c) > 0 \}$$

Note:

$j_f(c) \leq f(b) - f(a)$

Consider the subsets

$$D_1 := \{ c \in (a,b) \mid j_f(c) \geq f(b) - f(a) \}, \quad \# D_1 \leq 1$$

$$D_2 := \{ c \in (a,b) \mid j_f(c) \geq \frac{f(b) - f(a)}{2} \}, \quad \# D_2 \leq 2$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$D_k := \{ c \in (a,b) \mid j_f(c) \geq \frac{f(b) - f(a)}{k} \}, \quad \# D_k \leq k$$

Then, $D = \bigcup_{k=1}^{\infty} D_k$ hence is at most countable.

$\square$

Existence of inverse

Consider a cts $f : [a,b] \to \mathbb{R}$.

EVT $\Rightarrow$ $m := \inf_{x \in [a,b]} f(x)$, $M := \sup_{x \in [a,b]} f(x)$ are achieved

combine with IVT, $f([a,b]) = [m, M]$

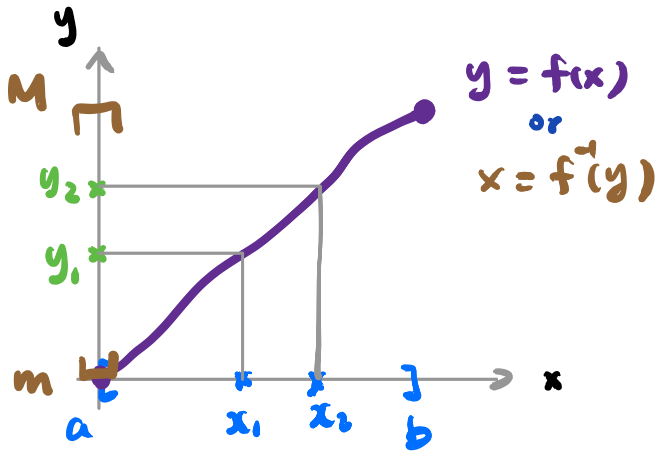
Q: When does the inverse $f^{-1} : [m, M] \to [a,b]$ exist?

Thm: If $f : [a,b] \to \mathbb{R}$ is strictly increasing & cts, then $f^{-1} : [m, M] \to [a,b]$ exists, and strictly increasing & cts.

"Sketch of Proof": By EVT and IVT, and f strictly increasing,

$f : [a,b] \to [m,M]$ is 1-1 and onto, so f^{-1} exists.

Claim: $f^{-1} : [m,M] \to [a,b]$ is strictly increasing.



Pf: Take any $y_1, y_2 \in [m,M]$ and $y_1 < y_2$.

Suppose $f(x_1) = y_1$, $f(x_2) = y_2$.

Note: $x_1 \neq x_2$.

Suppose $x_1 > x_2$. Since f is strictly increasing, we have

$$y_1 = f(x_1) > f(x_2) = y_2$$

Contradiction!

So, $x_1 < x_2$.

Claim: $f^{-1} : [m,M] \to [a,b]$ is cts

Pf of Claim: Suffices to check $\forall\, y_* \in (m,M)$,

$$\lim_{y \to y_*^-} f^{-1}(y) = \lim_{y \to y_*^+} f^{-1}(y)$$

Suppose NOT, then $\exists\, y_* \in (m,M)$ st $j_{f^{-1}}(y_*) > 0$.

ie $a \le \lim\limits_{y \to y_*^-} f^{-1}(y) < \xi < \lim\limits_{y \to y_*^+} f^{-1}(y) \le b$

fix some $\xi \neq f^{-1}(y_*)$ (*)

Let $f^{-1}(\tilde{y}) = \xi$. Note that $\tilde{y} \neq y_*$ by (*).

Case 1: $\tilde{y} < y_*$.

f^{-1} strictly increasing $\Rightarrow$ $\xi = f^{-1}(\tilde{y}) < f^{-1}(y_*)$

But previous thm $\Rightarrow$

$$\lim_{y \to y_*^-} f^{-1}(y) \overset{(*)}{<} \xi = f^{-1}(\tilde{y}) \le \sup_{y \in [m, y_*)} f^{-1}(y) = \lim_{y \to y_*^-} f^{-1}(y)$$

Contradiction!

Case 2: $\tilde{y} > y_*$ similar!

$\blacksquare$